Water and ice transfer in porous media near the phase transition point

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ABSTRACT: The mass transfer in the model frozen soil is studied theoretically. The porous medium consists of the permeable particles and has a regular texture. The mass transfer problem is solved in the following way: selecting the elementary cell in the medium, solving of the heat and mass task for this cell, averaging the resulting equations and transforming those different equations to the differential form.

The rates of the water and ice flows are sums of three forces: gradients of temperature, water pressure and ice stress. The explicit expressions are obtained for the transport coefficients, which are compared with each other. The experimental data [Horiguchi, Miller, 1980] are analyzed on the basis of the resulting equations.

1 INTRODUCTION

Equilibrium state of deformable frozen soil has three degrees of freedom what ordinarily are overburden, water pressure and temperature. The change of any parameter at boundary of the sample disturbs the balance and leads to the heat and mass transfer in porous medium (Gorelik et al. 2001). The water movement in this process is described by Darcy’s law only (Gilpin 1980) or in addition is taken into account the influence of temperature gradient (Nakano 1990). With regard to ice most researchers accept its immobility relative to the soil matrix. Few models assume the possibility of ice motion in a simplest form. For example, the soil matrix is not resisted to ice motion (O’Neill & Miller 1985) which may only be realised under special conditions. The theoretical study of the water and ice movement in the porous media model is needed to eliminate such ambiguities.

Freezing of homogenous clay soil induces its structural transformation (Ershov 1988). The ice inclusions that are formed in frozen soil near the freezing front take the water not only from external resources but also from the neighbouring unfrozen soil (cryosuction). The soil domains that feed the segregated ice are consolidated and the freezing point in these domains drops (Gorelik & Kolunin 1996). The pore distribution in the frozen soil changes in comparison with the unfrozen soil – the partitions of the small and large pores increase. As a result the unfrozen formations may be held in the frozen soil at negative temperatures near 0°C.

The water filtration through the unfrozen domains plays an important role in mass transfer in frozen soil, the total hydroconductivity of which is defined in many respects by the presence, configuration and properties of these domains. The rate of cryogenic processes both in nature and in laboratory experiments is frequently defined by the hydroconductivity of frozen soil (Gorelik et al. 1999).

The problem of water and ice transfer in porous media will be solved subsequently. At first the equations are derived to describe the behaviour of a permeable particle in ice. The developed approach is used for studying the ice movement through the set of the particle chains. As a result not only the mass transport equations for the frozen soil are determined but also the explicit form of transport coefficients is found.

2 A PARTICLE MOVEMENT IN ICE

Consider a porous spherical particle in an ice massif, with an imposed temperature gradient in the ice of $G$. The temperature at the object location is higher than the freezing point of water in the pores. An external force $F_{ex}$ what acts on the particle is parallel to the temperature gradient of $G$. Fix the spherical co-ordinate to the particle and place $Z$-axis in parallel with the force (Fig. 1).

Movement of the object occurs because of regelation phenomena (Nye 1967, Gilpin 1979). In front of the particle the ice melts, water migrates through porous

![Figure 1. Scheme of the fine-pored particle movement in ice.](image)
object to the rear side, where it freezes. The temperature field of this system is well known

\[ t(r, \theta) = \begin{cases} 
  t_0 + a r \cos \theta, & 0 \leq r \leq R \\
  t_0 + G r \cos \theta + \frac{b R^3}{r^2} \cos \theta, & r \geq R 
\end{cases} \]  

(1)

where \( R \) – radius of particle; \( t_0 \) – temperature at object centreline,

\[ a = \frac{3 G \lambda_i + V_i \kappa \rho_i}{2 \lambda_i + \lambda_p}, \quad b = \frac{(\lambda_i - \lambda_p) G + V_i \kappa \rho_i}{2 \lambda_i + \lambda_p} \]  

(2)

\( \rho_i \) – density of ice; \( \kappa \) – latent heat of fusion; \( \lambda_i, \lambda_p \) – thermal conductivities of ice and particle; \( V_i \) – velocity of ice relative to particle.

Liquid migrates in the porous object according to Darcy’s law: \( j = c \cdot \text{grad} \ p \). Therefore, the mechanical part of the task is reduced to the solution of Laplace’s equation for water pressure \( p \):

\[ \Delta p = 0 \]  

(3)

The continuity of the mass flows at the particle surface \( S_R \) gives a boundary condition:

\[ \frac{\partial p}{\partial n} \bigg|_{S_R} = -\frac{\rho_i V_i \cos \theta}{\rho_w c} \]  

(4)

where \( \rho_w \) – density of water; \( c \) – hydroconductivity coefficient of the particle.

The task (3)–(4) has the simple solution:

\[ p(r, \theta) = p_0 - \frac{\rho_i V_i}{\rho_w c} r \cos \theta \]  

(5)

\( p_0 \) – pressure of liquid at \( z = 0 \). The pressure distribution is axial symmetric and the streamlines are parallel to the axis \( Z \).

When the boundary condition at the particle surface is more complicated, the solution is not represented as a rule in analytical form. Further, the similar tasks will be solved by the approximate method named as “anisotropic hydroconductivity”. We now examine this method for the given system. Assume that the value of the hydroconductivity coefficient is different in parallel and perpendicular directions to \( Z \)-axis. Specify the polar co-ordinates respect to the particle. The point of origin coincides with the centre of the object. From now the letter \( r \) will be signified the radial coordinate in this system. In parallel direction the coefficient conductivity remains the same (real value \( c \)), in perpendicular direction that is assumed infinitely large. Therefore, the water pressure \( p \) and \( z \)-component of water flux \( j_z \) at any normal cross-section to axis \( Z \) don’t depend on radius \( r \).

The mass balance in the finite layer \( \Delta z \) (Fig. 2) is

\[ j_{z+\Delta z/2} S_{z+\Delta z/2} - j_{z-\Delta z/2} S_{z-\Delta z/2} + j_r \Delta S_R = 0 \]  

(6)

Turning to the differentials in the last relationship gives the equation for water flux \( j_z \):

\[ \frac{dj_z}{dz} = \frac{2z}{R^2 - z^2} \left( j_z - \frac{\rho_i V_i}{\rho_w} \right) \]  

(7)

Eliminating \( j_z \) from eq. (7) using Darcy’s law and solving the resulting equation gives the pressure distribution into the particle

\[ p(r, z) = p_0 - \frac{\rho_i V_i}{\rho_w c} r + K \ln \frac{R + z}{R - z} \]  

(8)

where \( K \) – constant. The finiteness of \( p(r, z) \) at \( z = \pm R \) requires \( K = 0 \).

Comparison of the equation (5) and (8) shows that the rigorous and approximate methods produce the identical relationships for the liquid pressure.

The unfrozen water film separates surfaces of ice and a particle. When the particle moves in ice the two forces act on its surface. First force of the disjoining pressure (Derjaguin et al. 1985) appears because of specific properties of water in the thin film and is perpendicular to interface. Second force of viscous friction is tangential to the surface and connected with liquid flow around the particle. When the size of the particle is significantly more than the layer thickness, the value of the second force may be neglected in comparison with the first (Gorelik et al. 2002). This supposition will be accepted in the future.

Thermodynamical equilibrium of water and ice near the particle surface is presented by the generalised Clapeyron-Clausius equation:

\[ \frac{P_l - P_0}{\rho_l} - \frac{p - P_0}{\rho_w} = -\frac{\kappa t}{T_0} \]  

(9)

Figure 2. Elements of selected particle layer.
where \( P_i \) – stress component in ice being perpendicular to the ice-water boundary; \( P_0 \) – reference pressure (atmospheric); \( t \) – temperature in °C and \( T_0 = 273.15 \) K.

The force \( dF_{pi} \) that acts on particle surface between cross-sections \( z \) and \( z + \Delta z \) (Fig. 2) is parallel to the axis \( Z \) and is expressed in term of ice stress \( P_i \):

\[
dF_{pi} = -2\pi R^2 P_i(\theta) \cos(\theta) \sin(\theta) d\theta \tag{10}
\]

Therefore, the total surface force is equal:

\[
F_{pi} = -2\pi R^2 \int_0^\pi P_i(\theta) \cos \theta \sin \theta d\theta \tag{11}
\]

The external force \( F_{ex} \) compensates this surface force:

\[
F_{ex} = 2\pi R^2 \int_0^\pi P_i(\theta) \cos \theta \sin \theta \ d\theta = 0 \tag{12}
\]

Substituting \( P_i \) from eq. (9) and \( t \) from eq. (1), \( p \) from eq. (5) at \( r/R \) into eq. (12) gives the velocity of the particle through ice \( V_p \):

\[
V_p = \frac{F_{ex}}{V_p} + \frac{3G\lambda}{2\lambda_i + \lambda_p} \frac{\kappa p_i}{T_0} \left( \frac{\rho_i}{\rho_p} \right)^2 \frac{1 + \frac{(\kappa p_i)^2}{T_0(2\lambda_i + \lambda_p)}}{c (\kappa p_i)^2} \tag{13}
\]

where \( V_p = (4/3)\pi R^3 \) – volume of particle. For example, in gravitational field \( F_{ex} = \rho_p V_p g \), where \( \rho_p \) – particle density; \( g \) – the acceleration of gravity.

3 WATER AND ICE TRANSPORT IN POROUS MEDIUM

Consider the porous medium, which is formed by fine-pored spherical particles (Fig. 3). The thickness of water film between ice and particle will be neglected. Ice intrudes partly between the neighbourhood particles leaving the clear space, which will be evaluated by an angle \( \theta_0 (0 < \theta_0 < \pi/2) \).

A filtration problem for a separate particle may be formulated as the task about the particle movement in ice with vertical channel (Fig. 4a). Streamlines aren’t parallel to axis \( Z \) i.e. there are radial mass flows inside the particle. The approximate method of «anisotropic hydroconductivity» will be used to solve filtration task in interval \(-z_0 \leq z \leq 0\). The equation (8) with additional boundary conditions

\[
p_i|_{z=-z_0} = p_1, \quad p_i|_{z=0} = p_2 \tag{14}
\]

gives the pressure distribution in the particle:

\[
p(r, z) = \frac{1}{c} \left\{ \frac{-P_i(r_0, z)}{\rho_w} - \left( \frac{p_1 - p_2 c - P_i(r_0, z)}{2} \right) \left( \ln \frac{R - z}{R + z} \right) + \frac{p_1 + p_2}{2} \right\} \tag{15}
\]

The cumulative liquid flow (m³/sec) through the particle is defined by liquid flux at section \( z = z_0 \):

\[
\Phi = \pi R^2 \sin^2 \theta_0 \left( -c \frac{\partial p}{\partial z} \bigg|_{z=z_0} \right) \tag{16}
\]

Substituting eq. (15) into eq. (16) gives

\[
\Phi = \pi R^2 \left( \frac{\rho_i}{\rho_p} V_{if}(\theta_0) - \frac{p_1 - p_2}{2R} \frac{1}{\ln[\tan(\theta_0/2)]} \right) \tag{17}
\]

where a function \( f(\theta) \) is

\[
f(\theta) = \sin^2 \theta + \frac{\cos \theta}{\ln[\tan(\theta_0/2)]} \tag{18}
\]

In particular the relationship (17) shows that the liquid flow through the particle is realised even if the pressure difference in the channel is equal to zero.
\( (p_1 = p_2) \) (Fig. 4b). Liquid that appears at boundary of particle near the channel outflows in it and the reversed process goes on the opposite side of the particle. Since the function \( f(\theta_0) \) is negative the direction of the flow \( \Phi \) is opposite to the velocity \( V_i \). The value of the flow depends on an angle \( \theta_0 \) and reaches maximum at \( \theta_0 = 20^\circ \).

By analogy with part 2 the particle acts on ice with the force \( F_{ip} \):

\[
F_{ip} = 2\pi R^2 \left( \frac{\rho_i}{\rho_w} \int_{\theta_0}^{\pi-\theta_0} p_R(\theta) \cos \theta \sin \theta \, d\theta - \frac{\kappa \rho_i}{T_0} \int_{\theta_0}^{\pi-\theta_0} \tau_0(\theta) \cos \theta \sin \theta \, d\theta \right)
\]

where \( \tau_0, p_R \) – temperature and water pressure at the particle surface. This force is induced by the interaction between the given particle and the nearest neighbours \( (F_{ix} = F_{ip}) \).

Thermal task is not solved here and temperature distribution at the surface of the particle in porous medium is accepted as for the isolated object.

Substituting eqs. (1) and (15) into the last equation and taking into account of \( z = R \cos \theta \) gives

\[
V_i = \frac{1}{D(\theta_0)} \left[ \frac{F_{ip}}{2\pi R^2} - \frac{\rho_i}{\rho_w} f(\theta_0) \frac{(p_2 - p_1)}{2R} - \frac{\kappa \rho_i}{T_0} \frac{2\lambda_i \cos^3 \theta_0 G}{2\lambda_i + \lambda_p} \right]
\]

where

\[
D(\theta) = \left( \frac{\rho_i}{\rho_w} \right)^2 \frac{1}{c} f_2(\theta) + \frac{2}{3} \frac{(\kappa \rho_i)^2}{T_0} \cos^3 \theta
\]

\[
f_2(\theta) = \frac{2}{3} \cos^3 \theta + \cos \theta f(\theta)
\]

Making the clear space between the particles to zero \( (\theta_0 \to 0) \) and using kinematics relationship \( V_i = -V_{ip} \), converts eq. (20) to eq. (13).

The force \( F_{ip} \) acts vertically and applies to the prominent parts (between attached particles) of ice column. Therefore the normal stress at horizontal section of ice is changed with coordinate \( z \). In the selected cell 1–2 (Fig. 3) the ice may be imagined as the cube with the cylindrical cavity of the radius \( R \sin \theta_0 \). When the vertical tangential force is applied to the cavity surface, the normal forces become various on the upper and lower faces of the cube. The lateral force \( F_{ip} \) is balanced by the difference of the normal stresses \( p_{i2} - p_{i1} \) at sections \( z = \pm R \):

\[
F_{ip} = \left( p_{i2} - p_{i1} \right) S_i
\]

where \( S_i \) – ice area at the section \( z = +R \) or \( z = -R \). The size of the cell is minimal at which the medium holds its properties as whole. The finite differences are converted to the differentials:

\[
\frac{\partial p_{i2}}{\partial z} = \frac{p_{i2} - p_{i1}}{2R}, \quad \frac{\partial p}{\partial z} = \frac{p_2 - p_1}{2R}
\]

The volumetric ice flux \( j_i \) through the porous medium is connected with the velocity \( V_i \):

\[
j_i = \left( \frac{S_i}{S_c} \right) V_i
\]

where \( S_c \) – area of horizontal face of the cell.

Substituting eqs. (23), (24) and (25) into (20) gives the law of the ice motion through the porous medium:

\[
j_i = -C_{ui} \frac{\partial p}{\partial z} - C_{iw} \frac{\partial p}{\partial z} - C_{ui} \frac{\partial t}{\partial z}
\]

with the transport coefficients

\[
C_{ii} = \frac{S_i^2}{\pi R^2 S_c} \frac{1}{D(\theta_0)}, \quad C_{iw} = \frac{\rho_i}{\rho_w} \frac{S_i}{S_c} f(\theta_0), \quad C_{ii} = \frac{\kappa \rho_i}{T_0} \frac{2\lambda_i}{2\lambda_i + \lambda_p} \frac{S_i}{S_c} f(\theta_0)
\]

The law of liquid transport in porous medium follows from eqs. (17), (20) and (24). Assuming volumetric water flux as \( j_w = \Phi/S_c \) gives

\[
j_w = -C_{iw} \frac{\partial p}{\partial z} - C_{iw} \frac{\partial p}{\partial z} - C_{iw} \frac{\partial t}{\partial z}
\]

where

\[
C_{iw} = \frac{\pi R^2}{S_c} \left[ \left( \frac{\rho_i}{\rho_w} \right)^2 \frac{f(\theta)}{D(\theta_0)} - \frac{c}{\ln \left( \tan(\theta_0/2) \right)} \right],
\]

\[
C_{iw} = \frac{\pi R^2 \kappa \lambda_i \cos^3 \theta_0}{\rho_w T_0} \frac{f(\theta_0)}{D(\theta_0)}
\]

It is interesting to remark that the coefficient \( C_{iw} \) is obtained equal to \( C_{iw} \).

The value of the angle \( \theta_0 \) may be estimated from the capillary properties of ice (Gilpin 1980). The curvature
$k$ of the ice surface at section A-A (Fig. 3) is:

$$k = \frac{1}{\eta_0} - \frac{1}{\eta_1}$$  \hspace{1cm} (30)

Water pressure $p$ and ice stress component $P_i$ are connected by Laplace’s equation:

$$P_i - p = \sigma_{nwk}$$  \hspace{1cm} (31)

where $\sigma_{nw}$ – surface tension of ice-water phase boundary.

Besides that the geometrical relationships take place (Fig. 3):

$$\tan \theta_0 = \frac{\eta_0}{R} \cdot R^2 + (\eta_0 + \eta_1)^2 = (R + \eta_1)^2$$  \hspace{1cm} (32)

The set of equations (9), (30) – (32) gives the explicit dependence of the angle $\theta_0$ on temperature, water pressure and particle size (Fig. 5). The graphs show that in the most interesting region of parameters the angle $\theta_0$ is in the interval $10^\circ$–$35^\circ$.

4 RESULTS

Let us call the coefficients $C_{kl}$ with $k = l$ as the direct coefficients and with $k \neq l$ as the cross coefficients. All calculation results were found at $S_i/S_c = 0.5$.

The ratio of the cross-to-direct coefficients depends weakly on the angle $\theta_0$ (Fig. 6) and in addition for ice does not depend on the hydroconductivity of the particle $c$.

The direct coefficient for water is by order of magnitude greater than the cross coefficients in region of $c > 2 \cdot 10^{-13} \text{m}^3 \cdot \text{s/kg}$ (Fig. 7). In this figure the thermal cross coefficient $C_{wtr}$ is replaced by the reduced coefficient $\bar{C}_{wtr} = (T_0/\rho_k)C_{wtr}$.

When the hydroconductivity coefficient $c$ is significant enough ($c > 10^{-13} \text{m}^3 \cdot \text{s/kg}$), the direct coefficient for water predominates over the direct coefficient for ice. The reversed situation is observed for small $c < 10^{-14} \text{m}^3 \cdot \text{s/kg}$ (Fig. 8).
The water and ice flows are interconnected and any one of them is given by three gradients: water pressure, ice stress and temperature.

In the general case the transport coefficients may be of the same order of magnitude. Therefore, mass transfer in porous media must be described by the general equations (26) and (28).

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