A LINEAR STABILITY ANALYSIS FOR THE INCEPTION OF DIFFERENTIAL FROST HEAVE

Rorik A. Peterson¹, William B. Krantz²

Institute of Arctic and Alpine Research
Department of Chemical Engineering
University of Colorado
Boulder, CO 80309-0450 U.S.A.

1. e-mail: Rorik Peterson@Colorado.edu
2. e-mail: Krantz@spot.colorado.edu

Abstract

A linear stability analysis was performed to determine whether one-dimensional frost heave has the propensity to evolve into multidimensional differential frost heave. Chena silt, Calgary silt and Illite clay, whose soil properties were obtained from literature data, were found to be unstable under a set of base conditions. Modeling the frozen layer as a purely elastic medium showed there is a most highly amplified wave number that is a function of the soil properties and environmental conditions. Sufficient surface load can completely stabilize the system. Increased frost penetration depth results in smaller wave numbers and, ultimately, stabilization. The most highly amplified wave numbers before stabilization correspond to wavelengths in the range of 2-10 meters which corroborates fairly well with field observations for patterned ground forms such as earth hummocks that are thought to arise from differential frost heave.

Introduction

Frost heave refers to an uplifting of the ground surface owing to freezing of water within the soil. Laterally nonuniform frost heave is referred to as differential frost heave (DFH). The latter can involve random heaving or can be in the form of regularly spaced earth mounds which constitute a form of patterned ground called hummocks (Williams and Smith, 1989; French, 1996). The magnitude of hummocks (1-10 meters) precludes their easy simulation in the laboratory; hence most research is focused on modeling. The model predictions can then be validated by field measurements made in the arctic and subarctic regions where hummocks are observed (Heginbottom; 1973, Mackay, 1977; Tarnocai and Zoltai, 1978). O’Neill and Miller (1985) first developed a rigorous model describing one-dimensional frost heave (i.e., ground surface defined as a plane) which recognized that there exists a region of ice/soil/unfrozen water below the frozen surface termed the frozen fringe. Fowler and Krantz (1994) later simplified this rather cumbersome model and replaced the rigid ice approximation used by O’Neill and Miller with a regelation mechanism that allowed for multidimensional frost heave. The resulting model consists of two coupled ordinary differential equations that describe the instantaneous location of the ground surface and the freezing front as functions of environmental conditions and soil properties. Lewis et al. (1993) and Fowler and Noon (1997) independently used this model to carry out a linear stability analysis (LSA) to ascertain whether the one-dimensional solution had any propensity to evolve into multidimensional (differential) frost heave. Whereas Lewis et al. (1993) concluded that one-dimensional frost heave was unstable (i.e., propensity for DFH to develop exists) under a wide range of conditions, Fowler and Noon (1997) predicted complete stability in the absence of snow cover. This paper attempts to resolve these conflicting predictions by considering realistic soil properties and properly treating the effect of the frozen layer on the stability of frost heave.

Approach

The approach to be pursued here is the development of a first-principles model that involves solving a coupled set of differential equations emanating from a continuum description of the relevant transport phenomenology and thermodynamics. This approach has been used successfully to describe salient features of patterned ground forms such as ice-wedge polygons (Lachenbruch, 1961), sorted polygons and stripes (Ray et al., 1983; Gleason et al., 1988) and sorted circles.
This approach compliments phenomenological and hierarchical modeling studies of patterned ground formation such as are being pursued by Hallet and coworkers (Werner et al., 1998) and others.

Figure 1 shows a schematic cross-section of water-saturated soil undergoing frost heave where \( z_s \) is the ground surface and \( z_f \) is the freezing penetration front. When Fowler and Krantz simplified the frost-heave model, they justified condensing the frozen fringe into a plane at \( z_f \). Thus, the resulting model equations are for the velocities \( V_s \) and \( V_f \) at the two surfaces \( z_s \) and \( z_f \), respectively. We carried out a LSA using these two equations. A LSA determines whether minute perturbations to the state variables will cause the system to move from the original solution to another solution due to a lower overall energy state. We perturb the model equations by substituting perturbed forms of each variable. These perturbed variables have the form \( \bar{X} + \varepsilon X' \) where the underlined variable is the one-dimensional base state and \( \varepsilon X' \) is an infinitesimally small perturbation. We then collect terms, cancel the base state, and neglect any terms of \( \varepsilon^2 \) and higher. This process results in two equations that relate the perturbed surface and freezing front locations to perturbations in all other variables.

In order to determine the perturbed temperature gradients within the frozen and unfrozen zones, we must obtain the perturbed temperature distribution. We solve the quasi-steady state, multidimensional Laplace equation (Bird et al., 1960) in both the frozen and unfrozen zones above and below the frozen fringe, respectively. The boundary conditions must also be perturbed. We looked at two different boundary conditions at the ground surface to evaluate various environmental conditions: constant temperature and Newton’s law of cooling for which the heat flux is proportional to the difference between the instantaneous surface and ambient temperatures.

According to normal mode analysis, each of the perturbations will have the form \( X_0 \exp(\Gamma t) \exp(i\alpha x) \) where \( \Gamma \) is a dimensionless growth-rate coefficient for the perturbation and \( \alpha \) is the dimensionless wave number \( (=2\pi d_o/l) \). One-dimensional frost heave will be unstable if any \( \alpha \) can be found that has a positive \( \Gamma \) (i.e., that will grow). After substituting this form for the per-

Table 1. Soil parameters and physical constants for the frost heave model

<table>
<thead>
<tr>
<th>Chena silt</th>
<th>Calgary silt</th>
<th>Illite</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( q )</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>1.6</td>
<td>0.7</td>
<td>6.4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ice density</th>
<th>Unfrozen water density</th>
<th>Heat of fusion</th>
<th>Heat capacity</th>
<th>Ice conductivity</th>
<th>Unfrozen water conductivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>900 kg/m³</td>
<td>1000 kg/m³</td>
<td>3x10⁵ J/kg</td>
<td>2 kJ/kg/K</td>
<td>4 W/m/K</td>
<td>2 W/m/K</td>
</tr>
</tbody>
</table>

Table 2. Base case conditions for LSA of the one-dimensional solution

<table>
<thead>
<tr>
<th>( z_f )</th>
<th>( d_o )</th>
<th>( P_L )</th>
<th>( T_S )</th>
<th>( T_B )</th>
<th>Top surface boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9 meters</td>
<td>1.0 meter</td>
<td>0 Pa</td>
<td>-10°C</td>
<td>0°C</td>
<td>Constant temperature</td>
</tr>
</tbody>
</table>
turbations, we arrive at two equations for the surface and freezing front perturbations as functions of \( \Gamma \) and \( \alpha \). The solution to Laplace’s equation in both the frozen and unfrozen regions then yields an explicit equation for \( \Gamma \) of the form

\[ A \Gamma^2 + B \Gamma + C = 0 \]

where \( A \), \( B \), and \( C \) are functions of \( \alpha \) and the environmental and soil parameters. Two solutions exist for this quadratic equation, an in-phase and an out-of-phase mode for all values of \( \alpha \).

The Fowler and Krantz frost-heave model requires two empirical functions which describe the soil: the cryostatic suction pressure, \( p_{\text{i}} - p_{\text{w}} \) where \( p_{\text{i}} \) and \( p_{\text{w}} \) denote the ice and water pressures, respectively, and the hydraulic conductivity, \( k_h \), as functions of the unfrozen water content, \( W \). While cryostatic pressure measurements are difficult to obtain directly, data for unfrozen water content as a function of supercooling temperature are available in the literature. By use of the Gibbs-Thompson equation (Burggraaf and Cot, 1996), one can relate the supercooling temperature to the cryostatic pressure. For this analysis, we used data presented by Horiguchi and Miller (1983) which include both the hydraulic conductivity and supercooling temperature as functions of unfrozen water content for three different soil types: Chena silt, Calgary silt, and Illite clay. Fowler and Krantz present the following form for the cryostatic suction pressure which displays the correct asymptotic behavior for both low and high unfrozen water contents.

\[
p_{\text{i}} - p_{\text{w}} = \frac{\left(1 - \frac{W}{\phi}\right)^p}{\left(\frac{W}{\phi}\right)^q}
\]

This function contains two empirical parameters, \( p \) and \( q \), that were obtained from Horiguchi and Miller’s data by a nonlinear least squares fit. The hydraulic conductivity has the form \( k_h = k_o (W/\phi)^\gamma \) where \( \phi \) is the porosity, \( k_o \) is the unfrozen soil hydraulic conductivity, and \( \gamma \) is an empirically determined parameter that was also obtained using a nonlinear least squares fit of Horiguchi and Miller’s data. These values for the three soil types and the other physical constants for the system are summarized in Table 1.

### Results and discussion

In order to study the effects of different soil types and environmental conditions on the stability of one-dimensional frost heave, a set of base conditions was established. These conditions were chosen to be typical of regions where DFH has been observed. A summary of these conditions is given in Table 2 where \( d_o \) is maximum freezing depth, \( P_L \) is surface load gauge pressure, \( T_S \) is the ground-surface temperature, and \( T_B \) is the temperature at the maximum frozen depth which is referred to as the basal plane.

Figure 2 shows \( \Gamma \) as a function of the wave number, \( \alpha \), for three different soil types under the base set of conditions. It is evident that under the base conditions, LSA predicts that all three soil types are unstable (i.e., a
range of $\alpha$ with positive $\Gamma$ s). It is also evident that the growth rate increases in magnitude with increasing wave number. Because the wave number is inversely proportional to the wave length, these results indicate that the most unstable wave for all soil types has an infinitesimally small wave length. While this result appears to be counterintuitive, it is not surprising. To help explain, Figure 3 shows a schematic of the system where the dashed lines indicate the initial, unperturbed one-dimensional solution. In part A we show an in-phase perturbation and in part B we show an out-of-phase perturbation, the only possible solutions. Before the perturbations occur, heat transfer occurs only in the vertical direction. However, once the system is perturbed, heat transfer can occur in both the $x$ and $z$ directions. To aid in visualizing the effect of perturbations on heat transfer, isotherms within the frozen layer have been drawn. In part A, it is evident that a net heat flux to the right will exist owing to lateral temperature gradients associated with the isotherms being closer together. Since heat is flowing from a crest region to a trough, we would expect retardation of the freezing in the trough region. Furthermore, it is evident that increasing wave number favors this type of heat transfer. This explains why the most highly amplified wave is an infinite wave number since the isotherms become vertical thereby resulting in maximum heat transfer in the $x$ direction for this limiting case.

In part B, the isotherms indicate that there would be heat transfer in both the positive and negative $x$ direction. In fact, to determine whether there is a net heat flux to or from a crest region, we must look at the relative magnitudes of the two perturbations, $\eta$ and $\xi$. However, this point is irrelevant because the out-of-phase mode is stable under all conditions for all soils. The reason for this unconditional stability will be discussed later.

Because DFH observed in nature, for example earth hummocks, has wave lengths on the order of meters, there must be some mechanism that prevents the wave length from becoming infinitesimally small; that is, there must be some restoring force which acts against bending of the frozen surface. Indeed, both Lewis et al. (1993) and Fowler and Noon (1997) recognized this. However, they chose to model the system differently. Lewis et al. (1993) modeled the frozen layer as a thin, purely elastic plate while Fowler and Noon (1997) modeled the frozen layer as a purely viscous medium. Literature on the mechanical behavior of frozen soils (Tsytovich, 1975) indicates that a viscoelastic model is probably most appropriate. Thus, real soils likely behave somewhere between the idealized models used by Lewis et al. (1993) and Fowler and Noon (1997). However, since LSA indicates only whether there is a propensity for growth of infinitesimal perturbations, it provides no information about how these perturbations will evolve in time. Thus, a model which provides an instantaneous response appears most appropriate for this type of analysis.

Using thin plate and shell theory (Brush and Almroth, 1975), we modeled the frozen layer as a thin, purely elastic plate with an elastic modulus, $E$, of 1000 kPa (Rajani and Morgenstern, 1994) as an additional base condition. Although the elastic modulus is a function of soil type, sub-freezing temperature, and water content
1000 kPa was chosen as a representative value for a typical frost-susceptible soil in the temperature range considered in this model (-10°C - 0°C). Figure 4 shows the effect of adding a restoring force on the stability of Chena silt for various values of $E$. We now see that there is a finite value for the most highly amplified wave number, $\alpha_{\text{max}}$, and that high wave numbers are now stabilized. This agrees with intuition because the force required to bend the frozen layer increases as the wave number increases. It is also interesting to note that $\alpha_{\text{max}}$ decreases as the elastic modulus increases. Physically, this means stiffer frozen soils will develop more closely spaced hummocks. Furthermore, there is not a critical value of the elastic modulus that will stabilize the system. As the modulus goes to infinity, $\Gamma$ at $\alpha_{\text{max}}$ approaches zero but remains positive.

We can observe the effect of surface load, $P_L$, on the stability of Chena silt as shown in Figure 5. Contrary to the effect of elastic modulus, we see that a finite value of load can stabilize the system. It is interesting to note that the critical value of load which will stabilize the system is only 30 kPa. Although this load is very small, DFH in the form of earth hummocks occurs when the only load is the weight of the overlying soil. It should further be pointed out that this result does not indicate that one-dimensional frost heave is suppressed above 30 kPa, only DFH. We observe similar behavior when we look at the effect of freezing depth on stability for Chena silt as shown in Figure 6. We see that as the freezing depth increases, $\alpha_{\text{max}}$ and the neutrally stable wave number decrease in magnitude. This means that as the freezing process progresses, the wave length of the most highly amplified wave number increases.

Figure 7 shows the effect of varying $h$, the overall heat-transfer coefficient, when the boundary condition has been changed to Newton’s law of cooling. In this instance, as $h$ increases, $\alpha_{\text{max}}$ decreases. Note that as $h \rightarrow \infty$, we approach the constant temperature boundary condition discussed earlier. We see that the Newton’s law of cooling boundary condition predicts shorter wave lengths. This boundary condition is also more appropriate because it best simulates a cold wind-swept ground surface such as that encountered in many arctic regions. Interestingly, Fowler and Noon (1997) found that this type of boundary condition was required for instability.

In addition to solving for $\Gamma = f(\alpha)$, we can determine $\Psi(\alpha) = \eta \xi^{-1}$, the ratio of the frozen front to the ground-surface perturbations. Figure 8 shows $\Psi$ as a function of $\alpha$ under the base conditions. Most notable is that $\Psi$ is positive for all values corresponding to in-phase perturbations. As mentioned earlier, the out-of-phase solution ($\Psi < 0$) is stable ($\Gamma < 0$) for all values of $\alpha$ under all conditions. We can explain this behavior by looking at the one-dimensional freezing problem.

In Figure 9, we plot the surface velocity, $V_s$, and freezing front velocity, $V_f$, as functions of surface load, $P_L$. 

(Tsytovich, 1975, p. 148)
for one-dimensional frost heave. Note that $V_f$ is negative because our coordinate system is measured positive upward. We see in region I that small increases in the surface load cause the surface velocity to increase while the freezing front velocity decreases in magnitude. This occurs because increases in load cause the ice content in the frozen fringe to also increase to support the extra load. The increase in ice content causes the cryostatic suction pressure driving force to also increase resulting in a greater flux of water upward to the warmest ice lens. Surface heave primarily occurs due to this upward flux of water forming ice lenses, thus, $V_s$ will also increase. However, the heat removal rate is a function of the temperature gradient in the frozen layer and not the surface load. Due to the increased flux of water to the warmest ice lens, $V_f$ will decrease in magnitude because less underlying saturated soil can now be frozen. Furthermore, if the load is increased beyond a critical value, $PL^*$, we see that $V_s$ decreases with load while $V_f$ increases in magnitude. Because the water flux is a balance between cryostatic suction pressure and frozen soil permeability, sufficiently high loads result in decreased water flux. It is these two processes, acting in opposite directions, that cause the counter-intuitive maximum in heave rate at $PL^*$. Noon (1996, p. 73) observed similar behavior in his analysis of secondary frost heave. Under high load conditions, the increased ice content has decreased the hydraulic conductivity dramatically. However, it is evident that under high or low load conditions the velocity behaviors are in-phase. Correspondingly, only in-phase perturbations can be unstable.

Conclusions

We have seen that the one-dimensional solution to the frost-heave model does have the propensity to evolve to multidimensional differential frost heave. This analysis represents an advance on the earlier LSA of Lewis et al. (1993) in that it properly accounts for the elasticity of the frozen layer. In contrast to the LSA of Fowler and Noon (1997), it predicts that one-dimensional frost heave can be unstable. We believe that Fowler and Noon (1997) did not use realistic soil properties and that they did not allow for the water permeation in their kinematic condition, i.e., in the condition that relates the velocity and amplitude at the mathematical plane defining the frozen fringe. It was shown that some soils have a greater propensity for DFH, and DFH can be suppressed with high surface loads. Modeling the frozen layer as a thin, purely elastic plate stabilizes high wave number solutions. When Newton’s law of cooling is used as a boundary condition for the ground surface, wavelengths similar to those observed in the field are predicted before the progressive freezing front penetration process completely stabilizes. The mechanism embodied in this LSA model may explain the origins of certain types of patterned ground such as earth hummocks, frost boils, and stone circles.

References


